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Asymptotic Approximation by Polynomials in the L_1 Norm

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1. The Chebyshev polynomials of the second kind, $U_n(x)$, play a role in L_1 approximation similar to that played by the Chebyshev polynomials of the first kind, $T_n(x)$, in L_∞ approximation. Thus the monic polynomial with minimum L_1 norm is $\bar{U}_n(x)$ just as the one with minimum L_∞ norm is $\bar{T}_n(x)$ where the bars indicate normalization and where we are considering the standard interval $I = [-1, 1]$. This analogy has not been extended to other situations in which the T_n are used to develop results in L_∞ approximation. In the present paper, we carry over the method of defect approximation [2, p. 98; 3] to the L_1 case and study expansions of entire functions in series of $U_n(x)$ to get asymptotic values of $E_n^1(f)$ where

$$E_n^1(f) = \min_{p_n \in \mathscr{P}_n} \| f - p_n \|_1 = \min_{p_n \in \mathscr{P}_n} \int_{-1}^{1} | f(x) - p_n(x) | \, dx$$

Here $f(x) \in C[-1, 1]$, the space of continuous functions of a real variable on I with real or complex values and $\mathscr{P}_n$ is the subspace of polynomials of degree $\leq n$ with complex coefficients. It is clear that an upper bound for $E_n^1(f)$ is given by $2E_n^\infty(f)$ where

$$E_n^\infty(f) = \min_{p_n \in \mathscr{P}_n} \| f - p_n \|_\infty = \min_{p_n \in \mathscr{P}_n} \cdot \max_{x \in I} | f(x) - p_n(x) |.$$

However, for the examples treated here, namely, $e^{\lambda x}$, λ complex, $\sin tx$ and $\cos tx$, t real, and e^{x^2}, the asymptotic value of $E_n^1(f)$ equals that of $E_n^\infty(f)$.

2. The expansion of $U_n(x)$ in powers of x is given by

$$U_n(x) = \sum_{j=0}^{[n/2]} (-1)^j \frac{n+1-2j}{n+1-j} \binom{n+1-j}{j} (2x)^{n-2j} \tag{1}$$

so that $\bar{U}_n(x) = 2^{-n} U_n(x)$. Further $\| \bar{U}_n \|_1 = 2^{1-n}$. If we denote the r-th derivative of $U_{n+2}(x)$ by $U_{n+2}^{(r)}(x)$, $r \geq 0$, then we have for any complex constant λ that

$$\sum_{r=0}^{n+2} \lambda^{-r} U_{n+2}^{(r)}(0) = \left(\frac{2}{\lambda}\right)^{n+2} \sum_{s=0}^{[(n+2)/2]} (-1)^s \left(\frac{\lambda}{2}\right)^{2s} \frac{(n+2-s)!}{s!} \tag{2}$$

Let now B be a continuous linear operator mapping $C[-1, 1]$ into itself and let the inverse operator B^{-1} exist and be continuous. By a standard compactness argument (see, e.g., [2, p. 1]) we are assured of the existence of $\tilde{Q}_{n+1} \in \mathscr{P}_{n+1}$ such that

$$\| B(\tilde{Q}_{n+1} - f) \|_1 \leq \| B(Q_{n+1} - f) \|_1$$

for all $Q_{n+1} \in \mathscr{P}_{n+1}$ and $f \in C[-1, 1]$. Now set $\delta_n = B(\tilde{Q}_{n+1} - f)$ and choose α_n so that

$$\tilde{Q}_{n+1}(x) - \alpha_n U_{n+1}(x) = p_n(x)$$

where $p_n(x) \in \mathscr{P}_n$. Then the following bounds hold for $E_n^1(f)$:

$$2 | \alpha_n | - \| B^{-1}\delta_n \|_1 \leq E_n^1(f) \leq \| p_n - f \|_1 \leq 2 | \alpha_n | + \| B^{-1}\delta_n \|_1 . \tag{3}$$

Proof. Since the right-hand inequality is obvious, it remains to prove the left-hand one. This follows from the elementary properties of the L_1 norm as follows:

$$\begin{aligned} E_n^1(f) &= E_n^1(\alpha_n U_{n+1} + f - \tilde{Q}_{n+1}) \geq E_n^1(\alpha_n U_{n+1}) - E_n^1(f - \tilde{Q}_{n+1}) \\ &\geq 2 | \alpha_n | - \| f - \tilde{Q}_{n+1} \|_1 = 2 | \alpha_n | - \| B^{-1}\delta_n \|_1 . \end{aligned}$$

3. In the application of this method of defect approximation, we shall take B to be a linear Volterra-type integral operator. By considering the equivalent differential equation and initial value condition, we can show the existence and continuity of B^{-1}. Furthermore, since in our examples, we have that B maps $\mathscr{P}_{n+1}$ into $\mathscr{P}_{n+k+1}$ where $k = 1$ or 2, we need only show that we can find a $\tilde{Q}_{n+1}(x) \in \mathscr{P}_{n+1}$ and a (complex) constant c_n such that $B(\tilde{Q}_{n+1} - f) = c_n U_{n+k+1}$. Our approach is constructive in that for some of the examples, polynomials p_n are determined which are almost best L_1 approximations in the sense that $\| p_n - f \|_1$ is asymptotically equal to $E_n^1(f)$.

EXAMPLE 1. Consider the function $f(x) = e^{\lambda x}$ where λ is a nonzero complex constant. $e^{\lambda x}$ is the unique solution of

$$Bf = f(x) - \lambda \int_0^x f(t)\,dt = 1. \tag{4}$$

We must now see if we can find a polynomial $\tilde{Q}_{n+1}(x) \in \mathscr{P}_{n+1}$ and a complex constant β_n such that

$$\tilde{Q}_{n+1}(x) - \lambda \int_0^x \tilde{Q}_{n+1}(t)\,dt - 1 = -\beta_n U_{n+2}(x). \tag{5}$$

For any given β_n, we have, by considering both real and imaginary parts, a total of $2n + 4$ linear equations in $2n + 4$ unknowns, with a nonsingular matrix. Hence we have a unique solution once we have determined β_n.

Repeated differentiation of (5) followed by a summation yields that

$$\tilde{Q}_{n+1}(x) = \beta_n \sum_{r=1}^{n+2} \lambda^{-r} U_{n+2}^{(r)}(x).$$

Substituting this polynomial into (5) gives

$$\beta_n = 1 \Big/ \sum_{r=0}^{n+2} \lambda^{-r} U_{n+2}^{(r)}(0). \tag{6}$$

From (2) we have the following estimate for β_n:

$$\frac{|\lambda|^{n+2}}{2^{n+2}(n+2)!}\left(1 - \frac{c}{n+2-c}\right) \leqslant |\beta_n| \leqslant \frac{|\lambda|^{n+2}}{2^{n+2}(n+2)!}\left(1 + \frac{c}{n+2-c}\right) \tag{7}$$

where $c = \exp(|\lambda|^2/4) - 1$. Hence $\beta_n \neq 0$ and $\tilde{Q}_{n+1}(x)$ is not identically zero. Differentiating (5) once yields

$$\tilde{Q}'_{n+1}(t) - \lambda \tilde{Q}_{n+1}(t) = -\beta_n U'_{n+2}(t). \tag{8}$$

In addition $\tilde{Q}_{n+1}(x)$ must satisfy the boundary condition

$$\tilde{Q}_{n+1}(0) - 1 = -\beta_n U_{n+2}(0).$$

Multiplying (8) by $e^{-\lambda t}$ and integrating from 0 to x, we find that

$$\tilde{Q}_{n+1}(x) - e^{\lambda x} = -\beta_n \left[U_{n+2}(x) + e^{\lambda x} \int_0^x e^{-\lambda t} U_{n+2}(t)\,dt\right] \tag{9}$$

Therefore $\| \tilde{Q}_{n+1}(x) - e^{\lambda x} \|_1 \leqslant 2 \mid \beta_n \mid K/\mid \lambda \mid$ where $K = e^{|\lambda|} + |\lambda| - 1$. If we now take $\alpha_n = 2(n+2)\beta_n/\lambda$, we have that

$$p_n(x) = \beta_n \left[\sum_{r=1}^{n+2} \lambda^{-r} U_{n+2}^{(r)}(x) - 2(n+2)\lambda^{-1} U_{n+1}(x) \right].$$

Referring to (3), we get the following inequality

$$2(|\beta_n|/|\lambda|)(2n+4-K) \leqslant E_n^1(e^{\lambda x}) \leqslant \| p_n - e^{\lambda x} \|_1 \leqslant 2(|\beta_n|/|\lambda|)(2n+4+K).$$

Substituting for $|\beta_n|$ from (7), we get our desired result,

$$E_n^1(e^{\lambda x}) = (|\lambda|^{n+1}/2^n(n+1)!)(1+O(n^{-1}))$$

which is the same asymptotic value as for $E_n^{\infty}(e^{\lambda x})$ [2, p. 96].

EXAMPLE 2. Consider now $f(x) = \cos tx$, $f(x) = \sin tx$, t real. The asymptotic deviation for these functions may be found by setting $\lambda = it$ in Example 1. The calculation from (4) to (9) is valid. In (6) β_n will be real for n even and imaginary for n odd. If we separate the real and imaginary parts of e^{itx} and set

$$\tilde{Q}_{n+1}(x) = \tilde{Q}_{n+1}^{[1]}(x) + i\tilde{Q}_{n+1}^{[2]}(x)$$

the $Q_{n+1}^{[1]}(x)$, $Q_{n+1}^{[2]}(x) \in \mathscr{P}_{n+1}$ are, respectively, even and odd for n even and odd and even for n odd. Thus for n even,

$$\beta_n = 1 \Big/ \sum_{r=0}^{(n+2)/2} (-1)^r t^{-2r} U_{n+2}^{(2r)}(0)$$

and for $f(x) = \cos tx$

$$p_{n-2}(x) = \beta_n \left[\sum_{r=1}^{(n+2)/2} (-1)^r t^{-2r} U_{n-2}^{(2r)}(x) + 4(n+2)(n+1) t^{-2} U_n(x) \right]$$

and

$$E_{2k}^1(\cos tx) = E_{2k+1}^1(\cos tx) = (|t|^{2k}/2^{2k-1}(2k)!)(1 + O(1/2k)),$$

while for $f(x) = \sin tx$,

$$p_{n-1}(x) = \beta_n \left[\sum_{r=0}^{n/2} (-1)^{r+1} t^{-2r-1} U_{n+2}^{(2r+1)}(x) + 2(n+2) t^{-1} U_{n+1}(x) \right]$$

and

$$E_{2k-1}^1(\sin tx) = E_{2k}^1(\sin tx) = (|t|^{2k+1}/2^{2k}(2k+1)!)(1 + O(1/2k)).$$

Again we have that E_n^1 is asymptotically equal to E_n^{∞}.

EXAMPLE 3. The function $f(x) = \exp(x^2)$ is the unique solution of the equation

$$Bf = f(x) - \int_0^x 2tf(t)\,dt = 1.$$

Hence we try to determine a real polynomial $\tilde{Q}_n(x)$ and a real β_n such that

$$\tilde{Q}_n(x) - \int_0^x 2t\tilde{Q}_n(t)\,dt - 1 = -\beta_n U_{n+2}(x). \tag{10}$$

We choose n to be even and determine β_n from the set of equations

$$\tilde{Q}_n^{(r)}(0) - 2(r-1)\,\tilde{Q}_n^{(r-2)}(0) = -\beta_n U_{n+2}^{(r)}(0), \qquad r = 2,\ldots, n+2.$$

We have that

$$2^{(n+2)/2}\left\{\prod_{s=0}^{n/2}(n+1-2s)\right\}\tilde{Q}_n(0)$$
$$= \beta_n\left[U_{n+2}^{(n+2)}(0) + \sum_{r=1}^{n/2} 2^r \prod_{s=0}^{r-1}(n+1-2s)\; U_{n+2}^{(n+2-2r)}(0)\right]. \tag{11}$$

But $\tilde{Q}_n(0) - 1 = -\beta_n U_{n+2}(0)$ and $2(n+1)\,2(n-1)\cdots 2.1 = (n+2)!/((n+2)/2)!$ so that we have that

$$\frac{(n+2)!}{\left(\frac{n+2}{2}\right)!} = \beta_n\left[U_{n+2}^{(n+2)}(0) + \sum_{r=1}^{(n+2)/2} 2^r \prod_{s=0}^{r-1}(n+1-2s)\; U_{n+2}^{(n+2-2r)}(0)\right].$$

On applying (1) we find that

$$\frac{1}{\left(\frac{n+2}{2}\right)!} = \beta_n\, 2^{n+2}[e^{-1/2}(1+O(n^{-1}))].$$

If $\tilde{Q}_n(x) = c_n x^n + \cdots$, then from (10), $2(n+1)!\,c_n = \beta_n 2^{n+2}(n+2)!$ so that $\alpha_n = 2(n+2)\,\beta_n$ and

$$2\alpha_n = e^{1/2}(1+O(n^{-1}))/2^{n-1}(n/2)!.$$

Solving the linear differential equation

$$\tilde{Q}_n'(t) - 2t\tilde{Q}_n(t) = -\beta_n U_{n+2}'(t)$$

with initial condition (11), we have that

$$\tilde{Q}_n(x) - e^{x^2} = -\beta_n \left[U_{n+2}(x) + e^{x^2} \int_0^x U_{n+2}(t)\, 2te^{-t^2}\, dt \right].$$

It follows that

$$\| \tilde{Q}_n(x) - e^{x^2} \|_1 \leqslant \beta_n \| U_{n+2} \|_1 \left[1 + \int_0^1 2xe^{x^2}\, dx \right] = 2K\beta_n$$

with $K < e$. Furthermore

$$2\beta_n[2(n+2) - K] \leqslant E^1_{2k-2}(e^{x^2}) = E^1_{2k-1}(e^{x^2}) \leqslant 2\beta_n[2(n+2) + K]$$

where $n = 2k$. Finally, we have that as $k \to \infty$

$$E^1_{2k-2}(e^{x^2}) = E^1_{2k-1}(e^{x^2}) = e^{1/2}(1 + O(1/2k))/2^{2k-1}k!\,.$$

(Cf. [5, p. 464]).

4. We now turn to the case where $f(x)$ is analytic in I and hence in a region containing I. Let E_R denote an ellipse with foci at ± 1 and sum of semiaxes $R > 1$. The set of functions analytic in E_R and Lebesgue integrable on E_R will be denoted by $A_R{}^1$. $f(z) \in A_R{}^1$ for some R.

LEMMA 1. *Let $f(z) \in A_R{}^1$ such that $f(z)$ is real for real z. For any $z_0 \in \operatorname{int}(E_R)$ we have that*

(i) *$f(z_0) = \alpha_0/2 + \sum_{n=1}^{\infty} \alpha_n U_n(z_0)$ where*

$$\alpha_n = \frac{2}{\pi} \int_{-1}^{1} f(x)\, U_n(x)\, (1 - x^2)^{1/2}\, dx, \qquad n = 0, 1, \ldots .$$

are the Fourier coefficients corresponding to the weight function $(1 - x^2)^{1/2}$.

(ii) *$|\alpha_n| \leqslant L(R)/R^{n+1}$ where*

$$L(R) = \frac{1}{\pi} \int_{E_R} |f(w)|\, |dw|.$$

Proof. (i) See [4, Theorem 9.1.1].

(ii) Setting $x = \cos t$, we have that

$$\alpha_n = \frac{1}{\pi i} \int_{-\pi}^{\pi} f(\cos t) \sin (n + 1)t \sin t\, dt.$$

Set $z = e^{it}$ and perform the integration with respect to z on the unit circle C, so that

$$\alpha_n = \frac{1}{2\pi i}\int_C f\left(\frac{z^2+1}{2z}\right)(z^{n-1} - z^{-n-1})\, d\left(\frac{z^2+1}{2z}\right).$$

By Cauchy's theorem

$$\begin{aligned}\alpha_n &= -\frac{1}{2\pi i}\int_{E_R} f\left(\frac{z^2+1}{2z}\right) z^{n+1} d\left(\frac{z^2+1}{2z}\right)\\ &\quad + \frac{1}{2\pi i}\int_{E_R} f\left(\frac{z^2+1}{2z}\right) z^{-n-1} d\left(\frac{z^2+1}{2z}\right).\end{aligned} \tag{12}$$

Now under the transformation $w = (1/2)(z + 1/z)$, w describes the ellipse E_R as z describes a circle of radius $1/R$ or R. Taking these cases respectively for the integrals in (12), we find that

$$|\alpha_n| \leq \frac{1}{\pi}\int_{E_R} |f(w)|\, |dw|/R^{n+1}. \tag{13}$$

COROLLARY 1. *If $f(z)$ satisfies the conditions of Lemma 1, then*

$$|E_n^1(f)| \leq 2L(R)/R^{n+1}(R-1).$$

Proof.

$$E_n^1(f) \leq \left\| f(x) - \frac{\alpha_0}{2} - \sum_{k=1}^{n} \alpha_k U_k(x) \right\|_1 \leq 2 \sum_{k=n+1}^{\infty} |\alpha_k|$$

from which the result follows.

THEOREM 1 (cf. [1, p. 115]). *Let $f(z)$ be an entire transcendental function which is real for real z. Then there exists a sequence of integers $n_1, n_2, \ldots$, such that*

$$\lim_{\mu\to\infty} (E^1_{n_\mu}(f)/2|\alpha_{n_\mu+1}|) = 1.$$

The sequence $\{n_\mu\}$ exists if and only if

(1) $\alpha_{n_\mu+1} \neq 0 \qquad \mu = 1, 2, \ldots,$

(2) $\sum_{r=n_\mu+2}^{\infty} |\alpha_r| = o(|\alpha_{n_\mu+1}|) \qquad \text{as} \quad \mu \to \infty.$

Proof. Since $f(z)$ is transcendental, we can find a sequence $\{n_p\}$ such that

$\alpha_{n_\mu+1} \neq 0$. Since by (13), $\lim_{n\to\infty}(\alpha_n)^{1/n} = 0$, there exists a subsequence $\{n_\mu\}$ such that for all μ

$$|\alpha_{n_\mu+k}/\alpha_{n_\mu+1}| \leqslant \delta_\mu^{k-1}, \qquad k = 1, 2, \ldots$$

where $\delta_\mu = o(1)$ as $\mu \to \infty$. Thus

$$\sum_{k=n_\mu+2}^{\infty} |\alpha_k| \leqslant \frac{\delta_\mu}{1-\delta_\mu} |\alpha_{n_\mu+1}|, \qquad \mu = 1, 2, \ldots . \tag{14}$$

Setting

$$p_{n_\mu}(x) = \alpha_0/2 + \sum_{k=1}^{n_\mu} \alpha_k U_k(x),$$

we have that

$$f(x) - p_{n_\mu}(x) = \sum_{j=1}^{\infty} \alpha_{n_\mu+j} U_{n_\mu+j}(x)$$

and

$$E_{n_\mu}^1(f) \leqslant \| f(x) - p_{n_\mu}(x) \|_1 \leqslant 2 | \alpha_{n_\mu+1} | + 2 \sum_{j=2}^{\infty} | \alpha_{n_\mu+j} |.$$

Furthermore

$$E_{n_\mu}^1(f) \geqslant E_{n_\mu}^1 \left(f - \sum_{j=2}^{\infty} \alpha_{n_\mu+j} U_{n_\mu+j} \right) - E_{n_\mu}^1 \left(\sum_{j=2}^{\infty} \alpha_{n_\mu+j} U_{n_\mu+j} \right)$$

$$\geqslant 2 | \alpha_{n_\mu+1} | - 2 \sum_{j=2}^{\infty} | \alpha_{n_\mu+j} |.$$

The result follows from (14).

EXAMPLE 1. If t is a real number, then [2, p. 96]

$$e^{tx} = I_0(t) + 2 \sum_{n=1}^{\infty} I_n(t)\, T_n(x)$$

where

$$I_n(t) = \sum_{j=0}^{\infty} \frac{(t/2)^{2j+n}}{j!\,(n+j)!}$$

is the modified Bessel function of order n. On differentiating, we obtain

$$e^{tx} = \frac{2}{t} \sum_{n=0}^{\infty} (n+1)\, I_{n+1}(t)\, U_n(x).$$

Now conditions (1) and (2) of Theorem 1 will be satisfied for $n_\mu = \mu$, $\mu = 1, 2, \ldots$. It follows that as $n \to \infty$

$$\begin{aligned} E_n{}^1(e^{tx}) &= 2 \cdot (2/|t|)(n+2)\, I_{n+2}(|t|)(1+o(1)) \\ &= (|t|^{n+1}/2^n(n+1)!)(1+o(1)) \end{aligned}$$

in agreement with our previous result.

EXAMPLE 2. From

$$\sin tx = 2 \sum_{n=0}^{\infty} (-1)^n\, J_{2n+1}(t)\, T_{2n+1}(x)$$

[2, p. 96], where

$$J_n(t) = \sum_{j=0}^{\infty} (-1)^j \left(\frac{t}{2}\right)^{2j+1} \Big/ j!\,(n+j)!$$

is the ordinary Bessel function of order n, we get by differentiating, that

$$\cos tx = \frac{2}{t} \sum_{n=0}^{\infty} (-1)^n\, (2n+1)\, J_{2n+1}(t)\, U_{2n}(x).$$

Taking n_μ as the sequence of odd integers, we find that

$$E_{2n}^1(\cos tx) = E_{2n+1}^1(\cos tx) = (|t|^{2n+2}/2^{2n+1}(2n+1)!)(1+o(1))$$

as before. A similar result holds for $\sin tx$.

EXAMPLE 3.

$$e^{tx^2} = \frac{a_0(t)}{2}\, U_0(x) + \sum_{n=1}^{\infty} a_n(t)\, U_n(x)$$

where

$$\begin{aligned} a_n(t) &= e^{t/2}[I_{n/2}(t/2) - I_{(n+2)/2}(t/2)] \\ &= (e^{t/2}(t/4)^{n/2}/(n/2)!)[1+o(1)]. \end{aligned}$$

Taking again n_μ as the sequence of odd integers, we find that

$$E_{2n}^1(e^{tx^2}) = E_{2n+1}^1(e^{tx^2}) = (e^{t/2} \mid t \mid^{n+1}/2^{2n+1}(n+1)!)(1 + o(1))$$

which agrees with our previous result for $t = 1$.

We close by stating two theorems without proof inasmuch as their proof is almost identical with the proof of the corresponding theorem for the L_∞ norm.

THEOREM 2 (cf. [1, p. 116]). *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire transcendental function, real for real z, such that* $\lim_{n\to\infty} n^{1/2} \mid a_n \mid^{1/n} = 0$. *Then there exists a sequence n_μ such that $a_{n_\mu+1} \neq 0$ and*

$$\lim_{\mu\to\infty} E_{n_\mu}^1(f) = \mid a_{n_\mu+1} \mid/2^{n_\mu}.$$

THEOREM 3 (cf. [2, p. 98]). *Let B be a continuous linear operator which maps the space $C[-1, 1]$ into itself and let the inverse operator B^{-1} exist and be continuous. Suppose that $f(x)$ is an entire function which is real for real z. Let $\tilde{Q}_{n+1}(x) \in \mathscr{P}_{n-1}$ be such that*

$$\| B(\tilde{Q}_{n+1} - f)\|_1 \leqslant \| B(Q_{n+1} - f)\|_1 \qquad \textit{for all} \quad Q_{n+1} \in \mathscr{P}_{n-1}$$

and let α_n be such that

$$\tilde{Q}_{n+1}^{(x)} - \alpha_n U_{n+1}(x) = p_n(x) \qquad \textit{for some} \quad p_n(x) \in \mathscr{P}_n .$$

Then there exists a sequence of integers n_μ, $\mu = 1, 2,\ldots$, such that

$$\| p_{n_\mu} - f\|_1 = E_{n_\mu}^1(f)(1 + o(1)) \qquad \textit{as} \quad \mu \to \infty.$$

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